PMM U.S.S.R., Vol.44, pp.70-75

0021-8928/81/1 0070 $7.50/0

UDC 532.72

THE STEFAN TYPE PROBLEM OCCURRING IN THE INVESTIGATION OF SALT DISSOLUTION AND TRANSPORT PROCESS IN SOIL *

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The boundary value problem associated with a parabolic equation whose coefficients become discontinuous from some instant of time along a beforehand unknown sliding line is considered. Such problem defines, for instance, the dissolution and removal of substances from the soil under hydraulic structure foundations, flushing of saliferous soils, etc.

A characteristic of the investigated processes is that they run in two stages. In the first, dissolution occurs throughout the filtration region, while the second is characterized by the emergence and increase with time of a zone of complete dissolution of the substance solid phase. Similar problems were considered earlier in /1—3/.

A numerical method using finite differences is proposed here. The existence and uniqueness of solutions of finite difference problems is proved, and an iteration process whose convergence is also proved is presented. Proof is also given of the convergence of solutions of finite difference problems to the solution of the input problem as pitch grid approaches zero, which confirms the existence of such solutions.

1. We pass to the formulation of the problem which for convenience we shall consider on the example of flushing saliferous soils. We examine a homogeneous layer of soil of depth L containing noxious salts in solid and fluid states. Let us assume that the flushing water which may also contain salts permeates into the layer from its surface. The concentration of salts in the water is assumed to be $c_* > 0$. The salt and filtration streams that take place under these conditions are one-dimensional and parallel to the x-axis with the soil surface taken as the reference plane. If at the initial instant of time the distribution of solid salts is such that their quantity increases with depth or remains constant, then, after some time from the commencement of flushing, a region of complete dissolution of salts is formed near the soil surface. That region is separated from the region containing solid phase salts by an a priori unknown shifting boundary $l(t)$, where t is the time. The unknown concentrations of dissolved and solid salts are defined by functions $c(x, t)$ and $N(x, t)$, respectively. Functions $c(x, t), N(x, t)$, and $l(t)$ are determined in conformity with /3/ by the following equations:

$$
\begin{aligned}
& m_2 c_t = Dc_{xx} - v(t)\, c_x - N_t \\
& N_t = -\gamma (c^* - c) \\
& 0 < x < L, \quad 0 < t \leqslant t^*; \quad l(t) < x < L, \quad t > t^* \\
& m_1 c_t = Dc_{xx} - v(t)\, c_x \\
& N(x, t) \equiv 0, \quad 0 < x < l(t), \; t > t^* \\
& c(x, 0) = c^*, \quad N(x, 0) = N_*(x) > 0 \\
& Dc_x(0, t) = v(t)(c(0, t) - c_*), \quad c_x(L, t) = 0, \; t > 0 \\
& c(l(t) - 0, t) = c(l(t) + 0, t), \quad c_x(l(t) - 0, t) = c_x(l(t) + 0, t) \\
& N(l(t) + 0, t) = 0, \; t > t^*, \quad t^* = \min\{t: N(0, t) = 0\}
\end{aligned}
\tag{1.1}
$$

where m_1 and m_2 are coefficients of soil porosity, D is the effective coefficient of diffusion, $v(t)$ is the filtration rate, γ is the coefficient of the dissolution rate, c^* is the saturation concentration, and c_* is the concentration of salts in the water permeating the soil. The subscripts x and t at the unknown functions denote derivatives with respect to respective variables.

For problem (1.1) which defines the first stage of the process $0 < t \leqslant t^*$ the following statement is valid /4,5/.

Lemma 1. If the assumptions

$$
\begin{aligned}
& m_2 > 0, \quad D > 0, \quad \gamma > 0, \quad c^* > c_* > 0, \quad v^* \geqslant v(t) \geqslant v_0 > 0 \\
& v(t) \in C^1[0, T], \quad v'(t) \geqslant 0, \quad N_*(x) \in C^1[0, L], \quad N_*'(x) \geqslant a > 0
\end{aligned}
$$

*Prikl.Matem.Mekhan., 44, No.1, 104-112, 1980 hold, then there exists a unique instant $t^* > 0$ such that for $0 < t \leqslant t^*$ a inique classical solution of problem (1.1) obtains, and the following estimates are valid:

$$c_* \leqslant c(x,t) < c^*;\quad 0 \leqslant c_x(x,t) \leqslant K_0;\quad 0 < a \leqslant N_x(x,t) \leqslant A$$
$$\alpha_0 \leqslant c_t(x,t) \leqslant 0$$

where A, K_0, and α_0 are constants dependent of the problem coefficients.

Below we examine only the second stage $(t > t^*)$, taking as the initial instant of time $t = t^*$ and functions $c_0(x) = c(x, t^*)$ and $N_0(x) = N(x, t^*)$ as the initial conditions. Note that the estimates of Lemma 1 are valid for these functions. We shall consider problem (1.1) on the above assumptions.

Definition. The set of three functions $c(x,t)$, $N(x,t)$, and $l(t)$, such that $c(x,t)$ and $N(x,t)$ are continuous in $Q_T = [0, L] \times [0, T]$ satisfy the equations and the right-hand boundary condition in (1.1) in the classical sense, and the left-hand side one in the sense that $\lim_{x\to 0} Dc_x(x,t) = v(t)(c(0,t) - c_*)$, will be called the solution of problem (1.1) for $t > t^*$. The condition of union of the derivative of c_x on the curve $l(t)$ is satisfied in the sense that

$$\lim_{\varepsilon\to 0} |\, c_x(l(t)-\varepsilon, t) - c_x(l(t)+\varepsilon, t)\,| = 0$$

Function $l(t) \in C[0, T]$ satisfies the equation $N(l(t), t) = 0$.

2. To prove the existence of a solution of this problem we, first, pass to the finite difference problem,and shall prove the existence and uniqueness of its solution and, then,show that the sequence of finite difference problems converges to the solution of the input problem (1.1) as the gird pitch approaches zero.

Let us consider the sequence of grids

$$w_p = \{x_i = ih_p,\quad t_j = \sum_{k=1}^{j} \tau_k,\quad i = 0, \ldots, M,\quad j = 1, \ldots, M\}$$

where τ_k is selected so as to conform to the method of catching the shifting boundary at a node of the grid /6/. We select an implicit four point scheme and obtain the following finite difference equation:

$$\begin{aligned}
&m_1 (c_i^j)_t = D(c_i^j)_{xx} - v^j (c_i^j)_x,\quad i = 1, \ldots, j-1\\
&m_2 (c_i^j)_t = D(c_i^j)_{xx} - v^j (c_i^j)_x + \gamma(c^* - c_i^j),\quad i = j+1, \ldots, M-1\\
&(N_i^j)_t = -\gamma(c^* - c_i^j),\quad i = j, \ldots, M\\
&D(c_1^j)_x = v^j (c_0^j - c_*),\quad (c_M^j)_x = 0,\quad c_i^0 = c_0(x_i),\quad i = 0, \ldots, M\\
&(c_j^j)_x = (c_{j+1}^j)_x,\quad N_j^j = 0,\quad j = 1, \ldots, M\\
&(c_i^j)_x = (c_i^j - c_{i-1}^j)/h,\quad (c_i^j)_{xx} = (c_{i+1}^j - 2c_i^j + c_{i-1}^j)/h^2,\quad (c_i^j)_t = (c_i^j - c_i^{j-1})/\tau_j
\end{aligned}\tag{2.1}$$

$$\tau_j = N_j^{j-1}/(\gamma(c^* - c_j^j)),\quad j = 1, \ldots, M \tag{2.2}$$

where the last equation was obtained from the equation $(N_j^j)_t = -\gamma(c^* - c_j^j)$ with allowance for condition $N_j^j = 0$.

For solving problem (2.1)—(2.2) we obtain a number of a priori estimates.

Lemma 2. Let

$$D > 0,\ m_1 > 0,\ m_2 > 0,\ \gamma > 0,\ c^* > c_* > 0,\ v^* \geqslant v^j \geqslant v_0 > 0$$

then the estimate

$$c_* \leqslant c_i^j \leqslant c^*,\quad i = 0, \ldots, M,\ j = 1, \ldots, M$$

is valid. The proof is based on the maximum principle /7/.

Lemma 3. Let the conditions of Lemma 2 be satisfied and, furthermore

$$v^j \geqslant v^{j-1},\quad j = 1, \ldots, M,\quad \gamma > v^* K_0/(c^* - c_M^0)$$
$$K_0 = (1 + v^* h/D)^M v^* c^*/D$$

then the estimates

$$(c_i^j)_t \leqslant 0,\ 0 \leqslant (c_i^j)_x \leqslant K_0,\ i = 0, \ldots, M,\ j = 1, \ldots, M$$

hold. We shall prove this by induction. From Lemma 1 we have

$$0 \leqslant (c_i^0)_x \leqslant K_0,\quad D(c_i^0)_{xx} - v^0 (c_i^0)_x + \gamma(c^* - c_i^0) \leqslant 0,\quad i = 1, \ldots, M-1$$

Let us prove that $(c_i^1)_t \leqslant 0$ by contradiction. Let

$$\max_{i=0,\ldots,M} (c_i^1)_t = (c_{i_0}^1)_t > 0$$

We shall consider the following cases:

1) $2 \leqslant i_0 \leqslant M-1$ when we obtain the following contradictory inequality:

$$0 < m_2 (c_{i_0}^1)_t = D (c_{i_0}^1)_{xx} - v^1 (c_{i_0}^1)_x + \gamma (c^* - c_{i_0}^1) \leqslant D (c_{i_0}^\circ)_{xx} - v^\circ (c_{i_0}^\circ)_x + \gamma (c^* - c_{i_0}^\circ) - (v^1 - v^\circ) (c_{i_0}^\circ)_x < 0$$

2) $i_0 = 1$, when from (2.1) we obtain

$$(c_1^\circ)_{xx} = -\tau_1 ((c_1^1)_t)_{xx} \geqslant 0$$

On the other hand, the inductive assumption and the conditions of the lemma imply that

$$D (c_1^\circ)_{xx} \leqslant v^\circ (c_1^\circ)_x - \gamma (c^* - c_1^\circ) \leqslant v^* K_0 - \gamma (c^* - \mathring{c}_M) < 0$$

which also yields a contradiction;

3) with $i_0 = 0$ and $i_0 = M$, using the boundary conditions, we again obtain a contradiction. Hence $(c_i^1)_t \leqslant 0$.

Let us now assume that for $j = 1, \ldots, n$ the unknown inequalities are satisfied, and prove their validity for $j = n + 1$.

First we shall prove the inequality $(c_i^n)_x \geqslant 0$. by contradiction. We assume

$$\min_{i=0,\ldots,M} (c_i^n)_x = (c_{i_0}^n)_x < 0$$

Then, using the inductive assumption and Eqs. (2.1), we arrive at a contradiction for any $i_0 \in [0 : M]$, which means that $(c_i^n)_x \geqslant 0$ for all $i = 0, \ldots, M$.

To prove the inequality $(c_i^n)_x < K_0$ we use the left-hand boundary condition in (2.1) which yields

$$(c_1^n)_x \leqslant v^* (c^* - c_*) / D$$

Moreover, from the inductive assumption we have

$$m_2 (c_i^n)_t = D [(c_{i+1}^n)_x - (c_i^n)_x] - v^n (c_i^n)_x + \gamma (c^* - c_i^n) \leqslant 0$$

from which directly follows that

$$(c_i^n)_x \leqslant (1 + v^* h / D)^M v^* c^* / D$$

Finally a proof similar to that of estimate $(c_i^1)_t \leqslant 0$ shows that $(c_i^{(n+1)})_t \leqslant 0$.

Lemma 4. If the conditions of Lemma 3 are satisfied and c_{1i}^n and c_{2i}^n are the respective solutions of problem (2.1) for $\tau_n = \tau_{1n}$ and $\tau_n = \tau_{2n}$, then when $\tau_{2n} > \tau_{1n}$, we have $c_{2i}^n \leqslant c_{1i}^n$, $i = 0, \ldots, M$.

To prove this we examine the remainder $z_i^n = c_{2i}^n - c_{1i}^n$.

Function z_i^n satisfies the following equations:

$$m_i z_i^n / \tau_{1n} + (c_{1i}^n)_t (\tau_{1n} / \tau_{2n} - 1) = D (z_i^n)_{xx} - v_2^n (z_i^n)_x - \gamma_i z_i^n - (v_2^n - v_1^n) (c_{1i}^n)_x$$

$$i = 1, \ldots, n-1, \quad m_i = m_1, \quad \gamma_i = 0$$

$$i = n+1, \ldots, M-1, \quad m_i = m_2, \quad \gamma_i = \gamma$$

$$(z_n^n)_{xx} = 0, \quad D (z_1^n)_x = v_2^n z_0^n + (v_2^n - v_1^n) (c_{10}^n - c_*), \quad (z_M^n)_x = 0$$

Using the indirect proof method and taking into account the estimates of Lemmas 2 and 3, we find that $z_i^n \leqslant 0,\ i = 0, \ldots, M$, hence $c_{2i}^n \leqslant c_{1i}^n$.

Lemma 5. If the conditions of Lemma 3 are satisfied, then the estimates

$$0 < H_0 \leqslant \tau_j / h \leqslant H^*, \quad j = 1, \ldots, M; \quad 0 < a \leqslant (N_i)_x \leqslant K_2, \quad i = 0, \ldots, M$$

$$N_j^{j+1} \geqslant ah > 0$$

where H_0, H^*, and K_2 are constants independent of τ_j and h, are valid.

Indeed, from the equation $(N_i^j)_t = -\gamma (c^* - c_i^j)$ we have

$$N_i^j - N_i^\circ = -\sum_{k=1}^{j} \tau_k \gamma (c^* - c_i^k) \tag{2.3}$$

$$(N_i^j)_x = (N_i^\circ)_x + \sum_{k=1}^{j} \tau_k \gamma (c_i^k)_x$$

Using Eq. (2 2) for τ_j we obtain

$$\frac{\tau_j}{h} = \left[(N_j^\circ)_x + \sum_{k=1}^{j-1} \tau_k \gamma (c_j^k)_x \right] \Big/ (\gamma (c^* - c_j^j)) \tag{2.4}$$

from which immediately follows the lower bound $\tau_j / h \geqslant a / \gamma c^*$. To obtain the upper bound τ_j / h we derive the following inequality:

$$\frac{\tau_j}{h} \leqslant \left(A + \gamma K_0 \sum_{k=1}^{j-1} \tau_k \right) \Big/ (\gamma (c^* - \mathring{c}_M))$$

which we write in the form

$$t_j \leqslant (1 + hK_0 / (c^* - c_M^\circ))\, t_{j-1} + hA / (\gamma (c^* - c_M^\circ))$$

Using this inequality we obtain the following estimates:

$$t_j \leqslant (1 + hK_0 / (c^* - c_M^\circ))^{j-1} A / \gamma K_0 \leqslant C_1$$

The necessary estimate now follow directly from equalities (2.3) and (2.4).

Lemma 6. If the conditions of Lemma 3 are satisfied and $(v^j - v^{j-1}) / \tau_j \leqslant P$, then there exists such $K_1 < 0$ independent of h and τ_j that

$$(c_i^j)_t \geqslant K_1, \quad i = 0, \ldots, M, \quad j = 1, \ldots, M$$

The proof is by induction. From Lemma 1 we have the estimate

$$m_2\alpha_0 \leqslant D (c_i^\circ)_{xx} - v^\circ (c_i^\circ)_x + \gamma (c^* - c_i^\circ), \quad \alpha_0 < 0$$

We shall prove that $(c_i^1)_t \geqslant \alpha_1$, where

$$\alpha_1 < \min \{(m_2\alpha_0 - PK_0\tau_1) / (m_2 + \gamma\tau_1), \quad -P (c^* - c_*) / v_0\}$$

Let us assume the opposite:

$$\min_{i=0,\ldots,M} (c_i^1)_t = (c_{i_0}^1)_t < \alpha_1$$

and first consider the case of $i_0 = 1$. We have $D (c_1^1)_{xx} = 0$ and $D (c_1^\circ)_{xx} - v^\circ (c_1^\circ)_x + \gamma (c^* - c_1^\circ) \geqslant m_2\alpha_0$. Hence

$$\tau_1 D (-(c_1^1)_t)_{xx} - v^\circ (c_1^\circ)_x + \gamma (c^* - c_1^\circ) \geqslant m_2\alpha_0 \tag{2.5}$$

$$(c_1^1)_t - (c_0^1)_t \geqslant h^2 (m_2\alpha_0 - \gamma c^*) / (D\tau_1)$$

On the other hand, from the boundary conditions in (2.1)we have

$$(c_1^1)_t - (c_1^\circ)_t = [v^1\tau_1 (c_0^1)_t + (v^1 - v^\circ) (c_0^\circ - c_*)] h / (D\tau_1) \tag{2.6}$$

From (2.5)—(2.6) for a fairly small $h > 0$ we obtain the inequality

$$0 \leqslant (m_2\alpha_0 - \gamma c^*) (h / \tau_1 - v^1 h^2 / D\tau_1) + (c^* - c_*) (v^1 - v^\circ) / \tau_1 + v^1\alpha_1 \leqslant$$
$$P (c^* - c_*) + v^\circ\alpha_1$$

However by virtue of the selection α_1 $P (c^* - c_*) + v^\circ\alpha_1 < 0$, i.e. the obtained inequality is constradictory. The case of $i_0 \neq 1$ also results in a contradiction. Hence $(c_i^1)_t \geqslant \alpha_1$.

Let us now assume that for all $k = 1, \ldots, n-1$ the inequality $(c_i^k)_t \geqslant \alpha_k$ is satisfied and show that the $(c_i^n)_t \geqslant \alpha_n$, where $\alpha_n < 0$ is chosen from the condition

$$\alpha_n < \min \left\{ \frac{m_2\alpha_{n-1} - PK_0\tau_n}{m_2 + \gamma\tau_n}, \quad -\frac{v^* K_0}{m_1}, \quad -\frac{2 (2\gamma c^* + v^* K_0)}{m_1 - m_2}, \quad \alpha_n^* + \frac{m_2\alpha_{n-1} - \gamma c^*}{D\tau_n} h^2 \right\}$$

$$\alpha_n^* < \min \{\alpha_{n-1} - PK_0\tau_n / m_1, \quad -P (c^* - c_*) / v_0\}$$

First we point out that $(c_i^n)_t$ cannot attain a local minimum smaller than α_n^* for $i = 0, \ldots, n-2$. This is proved by contradiction with allowance for the boundary conditions in (2.1).

Let us now show that $(c_i^n)_t \geqslant \alpha_n$ for all $i = 0, \ldots, M$.

Let us assume that

$$\min_{i=0,\ldots,M} (c_i^n)_t = (c_{i_0}^n)_t < \alpha_n$$

Since $\alpha_n < \alpha_n^*$, this minimum cannot by virtue of the above remark be attained for $i_0 = 0, \ldots, n-2$. Let us consider the remaining cases:

a) $i_0 = n-1$. We have $D (c_{n-1}^n)_{xx} = m_1 (c_{n-1}^n)_t + v^n (c_{n-1}^n)_x$ and $D (c_{n-1}^{n-1})_{xx} = 0$. Then because of the selection of $\alpha_n < 0$

$$0 \leqslant \tau_n D ((c_{n-1}^{n-1})_t)_{xx} = m_1 (c_{n-1}^n)_t + v^n (c_{n-1}^n)_x < m_1\alpha_n + v^* K_0 < 0$$

The obtained contradiction shows that this case is impossible.

b) $i_0 = n$. In this case $D (c_n^n)_{xx} = 0$ and $D (c_n^{n-1})_{xx} = m_2 (c_n^{n-1})_t + v^{n-1} (c_n^{n-1})_x - \gamma (c^* - c_n^{n-1})$ from which follows the inequality

$$0 \leqslant (c_{n-1}^n)_t - (c_n^n)_t \leqslant (-m_2\alpha_{n-1} + \gamma c^*) h^2 / (D\tau_n) \tag{2.7}$$

Let us prove that $(c_{n-1}^n)_t \leqslant (c_{n-2}^n)_t$ by assuming that $(c_{n-1}^n)_t > (c_{n-2}^n)_t$. Then from (2.7) we obtain

$$\min_{i=0,\ldots,n-2} (c_i^n)_t \leqslant (c_{n-2}^n)_t < (c_{n-1}^n)_t \leqslant (c_n^n)_t + (-m_2\alpha_{n-1} + \gamma c^*) h^2 / D\tau_n < \alpha_n^*$$

However, as previously shown, $(c_i^n)_t$ cannot attain for $i = 0, \ldots, n-2$ a minimum smaller than α_n^*. Hence the above assumption is false, and in fact $(c_{n-1}^n)_t \leqslant (c_{n-2}^n)_t$.

Then we have

$$D (c_{n-1}^{n-1})_{xx} = 0, \quad D (c_{n-1}^n)_{xx} = m_1 (c_{n-1}^n)_t + v^n (c_{n-1}^n)_x$$

By subtracting the first equality from the second we obtain

$$[(c_n^n)_t - (c_{n-1}^n)_t] D\tau_n / h^2 = [(c_{n-1}^n)_t - (c_{n-2}^n)_t] D\tau_n / h^2 + m_1 (c_{n-1}^n)_t + v^n (c_{n-1}^n)_x$$

from which with allowance for (2.7) and the inequality $(c_{n-1}^n)_t \leqslant (c_{n-2}^n)_t$ we obtain

$$m_2\alpha_n - \gamma c^* \leqslant m_2\alpha_{n-1} - \gamma c^* \leqslant [(c_n^n)_t - (c_{n-1}^n)_t] D\tau_n / h^2 \leqslant m_1\alpha_n + (-m_2\alpha_n +$$
$$\gamma c^*) m_1 h^2 / (D\tau_n) + v^* K_0$$

This inequality yields the following:

$$[m_1 - m_2 + m_1 m_2 h^2 / (D\tau_n)]\, \alpha_n \leqslant \gamma c^* + m_1 \gamma c^* h^2 / (D\tau_n) + v^* K_0$$

Taking into account the selection of α_n with a fairly small $h > 0$ we obtain a contradictory inequality. Hence $(c_i^n)_t > \alpha_n$, $i = 0, \ldots, M$, which shows the existence of constant $K_1 < 0$ such that $(c_i^n)_t \geqslant K_1$, $i = 0, \ldots, M$, $n = 1, \ldots, M$.

3. Let us now investigate the problem of existence of solution of problem (2.1)—(2.2).

Theorem 1. If the condition of Lemma 3 is satisfied, the problem (2.1)—(2.2) has a unique solution.

Proof. Let us first assume that τ_j, $j = 1, \ldots, M$ are known. The solution of problem (2.1)—(2.2) can then be obtained by the method of run-through in explicit form /7/, and such solution is unique an continuously dependent on τ_j. We shall now show that a τ_j that satisfies Eq. (2.2) exists. For this we examine function

$$\Phi(\tau) = \tau - N_j^{j-1} / (\gamma (c^* - c_j^j))$$

By virtue of Lemma 5 $\lim_{\tau\to 0} \Phi(\tau) < 0$ as $\tau \to 0$, and for fairly large $\tau > 0 - \Phi(\tau) > 0$. This with the continuity of function $\Phi(\tau)$ implies the existence of such τ_j^* that

$$\tau_j^* = N_j^{j-1} / (\gamma (c^* - (c_j^j)^*))$$

Let us prove the uniqueness of τ_j^*, $j=1, \ldots, M$. We set $j = 1$ and assume the existence of the following two solutions of problem (2.1)—(2.2): τ_1^*, c_{1i}^1, N_{1i}^1 and τ_1^{**}, c_{2i}^1, N_{2i}^1. For definites we set $\tau_1^* < \tau_1^{**}$.

Then taking into account Lemma 4 we obtain

$$0 < \tau_1^* - \tau_1^{**} = N_1^\circ / (\gamma (c^* - c_{11}^1)) - N_1^\circ / (\gamma (c^* - c_{21}^1)) =$$
$$N_1^\circ (c_{21}^1 - c_{11}^1) / (\gamma (c^* - c_{21}^1)(c^* - c_{11}^1)) \leqslant 0$$

We have thus obtained a contradiction. Since the case of $\tau_1^* > \tau_1^{**}$ also results in a contradiction, hence $\tau_1^* = \tau_1^{**}$. We similarly establish by induction that $\tau_j^* = \tau_j^{**}$, $j = 2, \ldots, M$. The theorem is proved.

Let us consider the question of numerical solution of problem (2.1)—(2.2) in more detail.

Let us consider the following iteration process:

$$m_1 (c_{in}^{(s)})_t = D (c_{in}^{(s)})_{xx} - v^n (c_{in}^{(s)})_x, \quad i = 1, \ldots, n-1$$
$$m_2 (c_{in}^{(s)})_t = D (c_{in}^{(s)})_{xx} - v^n (c_{in}^{(s)})_x + \gamma (c^* - c_{in}^{(s)}), \quad i = n+1, \ldots, M-1$$
$$D (c_{1n}^{(s)})_x = v^n (c_{0n}^{(s)} - c_*), \quad (c_{Mn}^{(s)})_x = 0, \quad (c_{nn}^{(s)})_{xx} = 0$$
$$(c_{in}^{(s)})_t = (c_{in}^{(s)} - c_{in-1}^{(s)}) / \tau_n^{(s)}, \quad (c_{in}^{(s)})_x = (c_{in}^{(s)} - c_{i-1n}^{(s)}) / h$$

The recurrent approximation of $\tau_n^{(s+1)}$ is determined using formulas

$$\tau_n^{(s+1)} = (a_{s+1} + b_{s+1}) / 2$$
$$A_s = N_n^{n-1} / (\gamma (c^* - c_{nn}^{(s)})) < \tau_n^{(s)}, \quad a_{s+1} = a_s, \quad b_{s+1} = \tau_n^{(s)}$$
$$A_s \geqslant \tau_n^{(s)}, \quad a_{s+1} = \tau_n^{(s)}, \quad b_{s+1} = b_s$$
$$\tau_n^{(0)} = N_n^{n-1} / \gamma c^*, \quad a_1 = N_n^{n-1} / \gamma c^*, \; b_1 = N_n^{n-1} / \gamma (c^* - c_{nn}^{(0)})$$

For a given $\tau_n^{(s+1)}$ function $c_{in}^{(s+1)}$ is computed by the run-through method.

Theorem 2. If conditions of Lemma 3 are satisfied, the considered here process converges to the solution of problem (2.1)—(2.2).

Proof. The existence of solution τ_n, c_i^n, $i = 0, \ldots, M$ follows from Theorem 1. Let us prove that $\tau_n \in [\tau_n^{(s)}, A_s]$. For definiteness we assume that $\tau_n \leqslant \tau_n^{(s)}$. We have

$$A_s - \tau_n = N_n^{n-1} / (\gamma (c^* - c_{nn}^{(s)})) - N_n^{n-1} / (\gamma (c^* - c_n^n))$$

Lemma 4 implies that $c_{nn}^{(s)} \leqslant c_n^n$, hence $A_s - \tau_n \leqslant 0$ or $A_s \leqslant \tau_n$. The case of $\tau_n \geqslant \tau_n^{(s)}$ is similarly considered. Thus, either $\tau_n^{(s)} \leqslant \tau_n \leqslant A_s$ or $A_s \leqslant \tau_n \leqslant \tau_n^{(s)}$. We shall now prove that for any $s \geqslant 1$, $\tau_n \in [a_s, b_s]$. Indeed, for $s = 1$

$$\tau_n^{(0)} = a_1 = N_n^{n-1} / \gamma c^* \leqslant N_n^{n-1} / (\gamma (c^* - c_n^n)) = \tau_n$$

On the other hand, since $\tau_n \geqslant \tau_n^{(0)}$ hence by virtue of Lemma 4

$$\tau_n - b_1 = \tau_n - \frac{N_n^{n-1}}{\gamma (c^* - c_{nn}^{(0)})} = \frac{N_n^{n-1} (c_n^n - c_{nn}^{(0)})}{\gamma (c^* - c_n^n)(c^* - c_{nn}^{(0)})} \leqslant 0$$

Thus $\tau_n \in [a_1, b_1]$.

Let us now assume that $\tau_n \in [a_s, b_s]$ and prove that $\tau_n \in [a_{s+1}, b_{s+1}]$. In fact, if $\tau_n \geqslant (a_s + b_s) / 2 = \tau_n^{(s)}$, then it follows from the above $A_s \geqslant \tau_n \geqslant \tau_n^{(s)}$ and we select $a_{s+1} = \tau_n^{(s)}$, $b_{s+1} = b_s$. Hence $\tau_n \in [a_{s+1}, b_{s+1}]$. If $\tau_n \leqslant (a_s + b_s) / 2$, then $A_s \leqslant \tau_n \leqslant \tau_n^{(s)}$ and we select $a_{s+1} = a_s$ and $b_{s+1} = \tau_n^{(s)}$. As previously, we have $\tau_n \in [a_{s+1}, b_{s+1}]$. Thus τ_n and $\tau_n^{(s)} \in [a_s, b_s]$ $s \geqslant 1$.

Since by definition $b_s - a_s \to 0$ as $s \to \infty$, hence $\tau_n^{(s)} \to \tau_n$, as $s \to \infty$, and since c_{in} is continuously dependent on $\tau_n^{(s)}$, then $c_{in}^{(s)} \to c_i^n$, $i = 0, \ldots, M$ as $s \to \infty$. The theorem is proved.

Theorem 3. If the conditions

$$m_1 > m_2 > 0,\ D > 0,\ v^* \geqslant v^j \geqslant v_0 > 0,\ 0 \leqslant (v^j - v^{j-1})/\tau_j \leqslant P$$

$$\gamma > v^* K_0 / (c^* - c_M^\circ),\ c^* > c_* > 0$$

are satisfied, there exists a unique solution of the input problem (1.1).

Proof. Let us examine the sequence of grids w_p, $h_p \to 0$ with $p \to \infty$. For each grid we extend the definition of functions c_i^j, $(c_i^j)_x$, $(c_i^j)_{xx}$, $(c_i^j)_t$, N_i^j, and $(N_i^j)_t$ linearly /8/ over all Q_T, and denote them, respectively, by $c^p(x, t)$, $y^p(x, t)$, $r^p(x, t)$, $s^p(x, t)$, $N^p(x, t)$, and $R^p(x, t)$.

In addition we define the broken line

$$l^p(t) = jh + (t - t_j)\,h/\tau_j,\ \ t_j \leqslant t \leqslant t_{j+1},\ \ j = 1, \ldots, M$$

Lemma 5 implies that $\{l^p\}$ is compact in $C[0, T]$, hence there exists a subsequence l^{p_k} which uniformly converges to $l^\circ(t) \in C[0, T]$.

Using the Bernshtein estimates /8/ we find that $\{c^p\}$, $\{y^p\}$, $\{r^p\}$ and $\{s^p\}$, are compact in regions $G_{1\delta} = \{(x, t) : \delta \leqslant x \leqslant l^\circ(t) - \delta,\ \delta \leqslant t \leqslant T\}$ and $G_{2\delta} = \{(x, t) : l^\circ(t) + \delta \leqslant x \leqslant L - \delta,\ \delta \leqslant t \leqslant T\}$, while $\{N^p\}$, $\{R^p\}$ is compact in $G_{2\delta}$, where $\delta > 0$ is any fairly small number. This implies the existence of subsequence of numbers which we also denote by p_k, such that c^{p_k}, y^{p_k}, r^{p_k}, and s^{p_k} converge, respectively, to some $c^\circ(x, t)$, $y^\circ(x, t)$, $r^\circ(x, t)$, and $s^\circ(x, t)$ at any internal point Q_T, $x \neq l^\circ(t)$, and uniformly in $G_{1\delta}$ and $G_{2\delta}$ for any fairly small $\delta > 0$. Similarly we have /8/

$$y^\circ(x, t) = c_x^\circ(x, t),\ r^\circ(x, t) = c_{xx}^\circ(x, t),\ s^\circ(x, t) = c_x^\circ(x, t)$$

The proof of uniform convergence of N^{p_k} and R^{p_k} to $N^\circ(x, t)$, and $N_t^\circ(x, t)$, $l(t) \leqslant x \leqslant L$ is similar. As in /8/, we obtain that $c^\circ(x, t)$ and $N^\circ(x, t)$ satisfy the respective differential equations and boundary conditions in (1.1).

It follows from Lemmas 2, 3, 5, and 6 that $\{c^p\}$ is compact in Q_T and, consequently, $\{c^{p_k}\}$ uniformly converges in Q_T to $c^\circ(x, t) \in C(Q_T)$, i.e. the first condition of junction on $l^\circ(t)$ is satisfied in the classical sense. From these lemmas we further have that N^{p_k} and R^{p_k} uniformly converge in $[l^\circ(t), L] \times [0, T]$ to $N^\circ, N_t^\circ \in C$. The sequence $\{y^p\}$ is compact in $C[0, L]$ for any fixed $t \in [0, T]$, which follows from Lemmas 3 and 6, and, consequently, $y^{p_k}(x, t_0)$ uniformly converges to $c_x(x, t_0)$ for any $t_0 \in [0, T]$. This also implies the fulfilment of the conjunction condition for c_x° on $l^\circ(t)$. Finally, taking into account the condition $N_j^j = 0$ and the convergence of $\{N^{p_k}\}$ and $\{l^{p_k}\}$, we obtain $N^\circ(l^\circ(t) + 0, t) = 0$.

Thus the set of three functions $c^\circ(x, t)$, $N^\circ(x, t)$, and $l^\circ(t)$ is the solution of problem (1.1). It uniqueness is proved by the method of contractive mappings. The theorem is proved.

The authors thanks F. P. Vasil'ev for advice and interest in this work.

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Translated by J.J.D.